On the additivity of generation-recombination spectra.
Part 2: $1/f$ noise

F.N. Hooge*

Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513 5600, MB Eindhoven, The Netherlands

Received 25 November 2002; received in revised form 31 January 2003

Abstract

The McWhorter model explains $1/f$ noise as an addition of generation-recombination spectra. Here we investigate whether addition of spectra is allowed. A condition for additivity is derived. If that condition is not met, then the GR spectra do not add together but they mix; resulting in one Lorentzian with $\tau_L$ given by $1/\tau_L = \sum 1/\tau_i$. A $1/f$-like spectrum appears at very low frequencies. A numerical estimate for the $1/f$ noise parameter $\alpha$ follows from our analysis. The disagreement with experimental values found with MOSTs makes it very unlikely for the McWhorter model to be correct.

© 2003 Elsevier Science B.V. All rights reserved.

PACS: 72.70+m

Keywords: Generation-recombination noise; $1/f$ noise; McWhorter model

1. Introduction

In part 1 we analyzed the generation-recombination noise of a semiconductor with two kinds of recombination centers [1]. We found that two situations may occur:

(i) Addition: the noise spectrum is the sum of two GR spectra

$$S = S_A + S_B,$$

where $S_A$ is the GR spectrum of a sample where only $A$ is present (in the same amount as in the $(A + B)$ sample). $S_B$ is the spectrum of a sample with only $B$.

(ii) Mixing: the spectrum is one simple Lorentzian with $\tau$ given by

$$\frac{1}{\tau} = \frac{1}{\tau_A} + \frac{1}{\tau_B}.$$  

The characteristic time $\tau_A$ follows from a model in which the number of $A$ centers is $A$. The number of occupied centers, or the number of trapped electrons, is $a$. There is recombination $R_A$ from the $n$ free electrons (in the conduction band with $N$ states) to the $(A - a)$ empty centers and generation $G_A$ of free electrons from the occupied levels into the conduction band. The result is

$$\theta_A = \frac{1}{\tau_A} = G_A \left( \frac{1}{n^*} + \frac{1}{n} \right),$$

where $n^*$ is the density of states.
where $\alpha^*$ is defined by
\[
\frac{1}{\alpha^*} = \frac{1}{\alpha} + \frac{1}{A - a} \tag{4}
\]

In part 2 we will consider situations with many levels. We will not discuss all possible situations but restrict ourselves to one problem: under which condition do many GR spectra produce a $1/f$ spectrum? In 1939 Surdin [2] already pointed out that $1/f$ spectrum is obtained if the GR spectra have a wide range of $\tau$ values with statistical weight
\[
g(\tau) \, d\tau \propto \frac{1}{\tau} \, d\tau. \tag{5}
\]

Then
\[
S_x \propto \int_0^\infty g(\tau) \frac{4\tau(\Delta x)^2}{1 + \omega^2 \tau^2} \, d\tau
= \frac{4(\Delta x)^2}{\omega} \int_0^\infty \frac{1}{1 + (\omega\tau)^2} \, d(\omega\tau)
= \frac{4(\Delta x)^2}{\omega} \arctan(\omega\tau) \bigg|_0^\infty
= \frac{4\pi}{\omega} \frac{(\Delta x)^2}{\tau}\left(1 - \frac{\tau}{f}\right). \tag{6}
\]

Surdin’s proposal works if
(i) the individual GR spectra can be added together,
(ii) the individual spectra have the same variance $(\Delta x)^2$.

2. The McWhorter model

In 1957 McWhorter [3] proposed a physical model that fulfils the mathematical condition (5). He considered a semiconductor crystal covered by an oxide layer that is homogenously doped with one kind of traps. The probability that a free electron from the semiconductor reaches a trap at a distance $s$ from the interface by tunneling is
\[
P(s) = P(0)e^{-s/\lambda}, \tag{7}
\]
where $\lambda$ is the effective penetration depth. The probability of its being trapped at $s$ is proportional to $P(s)$, as all traps have the same properties. Therefore,
\[
\frac{1}{\tau_s} = \theta(s) = \theta(0)e^{-s/\lambda} \tag{8}
\]
because of the homogenous doping, $dX/ds = C$, where $X$ is the number of traps, we find
\[
g(\tau) = \frac{dX}{d\tau} = \frac{dX}{ds} \frac{ds}{d\tau} = C\frac{\lambda}{\tau}. \tag{9}
\]

This distribution of trapping times, based on tunneling into an oxide layer, results in a $1/f$ spectrum as is shown by (6). Few people have paid attention to the problem of adding together spectra, and to the use of the integral in Eq. (6). For a survey of this problem, see Ref. [4]. McWhorter himself suggested that the free electrons in a small volume near the interface can only interact with the adjacent small area of the oxide layer. If the area contains one trap, it will have one value of $\tau$. Different areas have different $\tau$’s because the distances of the traps to the interface are different.

In this paper we will investigate whether the addition of GR spectra is possible in a model where all traps are accessible to the electrons in the conduction band (see Fig. 1).

No details of the tunneling transition from the conduction band to a trap are considered here. The traps are of the same chemical nature, so their energy levels are at the same height. The tunneling is accounted for by different capture times: the further away from the interface, the slower the trap.

Fig. 2 shows relation (8). Instead of a continuous distribution we use a step function to facilitate the calculations. The traps are equidistant with an interval $\xi$. We enumerate the traps with $r$; $r_A = 1$, $r_B = 2$, etc.
\[
s_i = r_i\xi, \tag{10}
\]
we introduce
\[
\eta = \frac{\xi}{\lambda} \tag{11}
\]
so that (8) can be rewritten as
\[
\theta(s) = \theta(0)e^{-s/\lambda} = \theta(0)e^{-\eta}. \tag{12}
\]

We will use $r$ and $\xi$ to make simple mathematical approximations. They do not have a physical
meaning, in the way $\theta$ and $\lambda$ do. We are free to choose the widths of the intervals. The accuracy may depend on the value of $\xi$, but our final results should not contain $r$, $\xi$ or $\eta$.

3. Variances and correlations

We now investigate the variances and correlations, without considering the kinetics of the transitions. This is allowed since variances and correlations are equilibrium properties. We use the following notation:

$X_i$ for $A$, $B$, etc., the number of the states of the $A$, $B$, etc., centers. $i = 1$ for $A$, $i = 2$ for $B$ etc., $i = r_k$ for $K$.

$x_i$ denotes $a$, $b$, etc., the number of electrons in $A$, $B$, ...

In the McWhorter model

$A = B = C = \cdots$ $a = b = c = \cdots$

$a^* = b^* = c^* = \cdots$ ...

(13)

In part 1 we considered two extreme situations: $n \ll a^*, b^*$ and $n \gg a^*, b^*$. There was a transition range $b^* \ll n \ll a^*$. Now we treat the McWhorter model, where there is no such transition range because of (13). We analyze two extreme situations: mixing if $n \ll \sum x_i$, and addition if $n \gg \sum x_i$.

The variances are given by [5]

$$\frac{1}{\langle (\Delta x_i)^2 \rangle} = \frac{1}{x_i^*} + \frac{1}{n + \sum_{j \neq i} x_j^*}.$$  

(14)

$$\frac{1}{\langle (\Delta n)^2 \rangle} = \frac{1}{n} + \frac{1}{\sum x_i^*}.$$  

(15)

In both situations, mixing and addition, we find

$$\langle (\Delta x_i)^2 \rangle = x_i^*.$$  

(16)

Eq. (15) gives different results for the variance of $n$

If $n \ll \sum x_i^*$, $\langle (\Delta n)^2 \rangle = n,$

(17)

If $n \gg \sum x_i^*$, $\langle (\Delta n)^2 \rangle = \sum x_i^*.$

(18)
The discussion of the correlations in the fluctuations starts with the condition
\[ -\Delta n = \sum \Delta x_i. \]  

(19)

This relation does not imply anything for the donor or acceptor character of the centers. See Eq. (13) of Part 1 [1]. It only indicates that the total number of free electrons and electrons trapped in the centers is constant. Squaring (19) leads to
\[ \langle (\Delta n)^2 \rangle = \sum \langle (x_i)^2 \rangle + \sum_i \sum_{j \neq i} \langle \Delta x_i \Delta x_j \rangle. \]  

(20)

(1) If \( n \ll \sum x_i \) (16), (17) and (20) give
\[ n = \sum x_i^* + \sum \sum \langle \Delta x_i \Delta x_j \rangle, \]  

(21)

which means that \( \sum x_i^* \) and \( \sum \sum \langle \Delta x_i \Delta x_j \rangle \) nearly have the same value. There are considerable correlations; simple addition of the spectra \( S_A, S_B, \) etc. is impossible. Naive application of the McWhorter model is not allowed if \( n \ll \sum x_i \).

(2) \( n \gg \sum x_i^* \) if the McWhorter model works correctly there should be no correlations between any \( \Delta x_i \) and \( \Delta x_j \). This requires \( \sum \sum \langle \Delta x_i \Delta x_j \rangle = 0 \). Then (20) leads to \( \langle (\Delta n)^2 \rangle = \sum \langle (x_i)^2 \rangle \). The right-hand term equals \( \sum \Delta x_i^* \) according to (16). So (18) \( \langle (\Delta n)^2 \rangle = \sum \Delta x_i^* \) is valid. Then (15) requires \( n \gg \sum \Delta x_i^* \). Only under this condition does the McWhorter model produce a 1/f spectrum. The results of Section 3 are summarized in Table 1.

### Table 1

<table>
<thead>
<tr>
<th>( n \ll \sum x_i^* )</th>
<th>( n \gg \sum x_i^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle (\Delta x_i)^2 \rangle = x_i^* )</td>
<td>( \langle (\Delta x_i)^2 \rangle = x_i^* )</td>
</tr>
<tr>
<td>( \langle (\Delta n)^2 \rangle = n )</td>
<td>( \langle (\Delta n)^2 \rangle = \sum x_i^* )</td>
</tr>
<tr>
<td>( \sum \sum \langle \Delta x_i \Delta x_j \rangle \neq 0 )</td>
<td>( \sum \sum \langle \Delta x_i \Delta x_j \rangle = 0 )</td>
</tr>
</tbody>
</table>

4. Kinetic description

In analogy to part 1 (Section 4) we will analyze the noise “from \( K \)”. This means that we consider the generation-recombination processes between a center \( K \) and the band. The fluctuations created in \( K \) and in the band will die out and partly redistribute themselves over the other centers. Since the noise from other sources is analogous to the noise from \( K \), the final result is easily found for the individual sources “from \( A \)” to “from \( Z \)”.

In a very short time interval \( \delta t \), \( \rho \delta k \) electrons arrive in \( K \) by the recombination \( R_K \),
\[ \rho \delta k = R_K \delta t. \]  

(22)

The number of electrons leaving by \( G_K \) is
\[ \rho \delta k = G_K \delta t. \]  

(23)

The processes \( G_K \) and \( R_K \) could change the number of electrons present in \( K \), \( k \delta k \),
\[ k \delta k = R_K \delta t - G_K \delta t. \]  

(24)

We reserve the symbol \( \delta \) for quantities related to \( \delta t \). Fluctuations are introduced by \( S_A R_K \), the shot noise in \( R_K \) and by \( S \Delta G_K \), the shot noise in \( G_K \). Eq. (24) and Eqs. (22) and (23) then give
\[ \Delta(k \delta k) = S \Delta R_K \delta t - S \Delta G_K \delta t = \Delta(k \delta k) - \Delta(\rho \delta k). \]  

(25)

Since \( \Delta(k \delta k) \) and \( \Delta(\rho \delta k) \) result from Poissonian processes, and since they are uncorrelated it follows that
\[ [\Delta(k \delta k)]^2 = [\Delta(k \delta k)]^2 + [\Delta(\rho \delta k)]^2 \]
\[ = \rho \delta k + \rho \delta k = \rho \delta t + G \delta t = 2R_K \delta t. \]  

(26)

The decay of \( \delta k \) begins with the generation and recombination processes between \( K \) and the band. In a later stage processes from the band to other levels may become important. For the moment we assume that the other processes have little influence on the lifetime of the electrons in \( K \). We can then calculate the variance \( [\Delta(k \delta k)]^2 \). The variance is averaged over time for an extended period, \( T \), and consists of \( \delta t / \delta t \) uncorrelated, and decaying contributions \( 2R \delta t \) (26). Each contribution decays with \( \tau_K \) (\( \tau_K \ll T \)).

\[ \langle [\Delta(k \delta k)]^2 \rangle = \frac{T}{\delta t} \times \int_0^T [\Delta(k \delta k) e^{-t/\tau_K}]^2 \ dt \]
\[ = \frac{T}{\delta t} \times \int_0^T 2R_K \delta t e^{-2t/\tau_K} \ dt \]
\[ = \frac{T}{\delta t} \frac{1}{2} 2R_K \delta t \frac{\tau_K}{2} = R_K \tau_K. \]  

(27)
We now consider the initial decay of $\delta k$ in more detail.

$$\frac{d\Delta k}{dt} = -\Delta G_K + \Delta R_K.$$  \hspace{1cm} (28)

The symbol $\Delta R$ in Eq. (28) differs from $s\Delta R$ in Eq. (25). In Eq. (25) $s\Delta R$ means the shot noise in the current $R$, which is determined by $\bar{n}$ and $\bar{k}$. The source $s\Delta R$ causes the fluctuations $\Delta k$. In Eq. (28) $\Delta R$ means the fluctuations in $R$, caused by fluctuations in the numbers $k$ and $n$. The transition rates are proportional to the number of electrons in the initial state and the number of empty positions in the final state. The proportionality constants $\gamma$ are for the generation and $\rho$ for the recombination.

$$\bar{G}_K = \gamma N k \Delta G_K = \bar{G}_K \frac{\Delta k}{k}$$

$$\bar{R}_K = \rho n (K - k),$$

$$\Delta R_K = \bar{R}_K \left[ \frac{\Delta (K - k)}{K - k} + \Delta n \right]$$

$$= -\bar{R}_K \frac{\Delta k}{K - k} + \bar{R}_K \frac{\Delta n}{n}.$$  \hspace{1cm} (29)

$$\Delta R_K$$ has a $-\Delta k$ component. We add this $-\Delta k$ component to $\Delta G_K$. Eq. (28) then becomes

$$\frac{d\Delta R_K}{dt} = -\bar{G}_K \left[ \frac{\Delta k}{k} + \frac{\Delta k}{K - k} \right] + \bar{R}_K \frac{\Delta n}{n}$$

$$= -\bar{G}_K \frac{\Delta k}{k^*} + \bar{R}_K \frac{\Delta n}{n}.$$  \hspace{1cm} (30)

In the last step we use the definition of $k^*$, according to Eq. (4). Eq. (31) shows that there are two contributions in the variations of $\Delta k$, one from $\Delta k$ and one from $\Delta n$. For Boltzmann statistics the $\Delta k$ and $\Delta n$ contributions correspond to $\bar{G}_K$ and $\bar{R}_K$; for Fermi statistics this is not the case. In the following we shall use the $\Delta k$, $\Delta n$ description and not the $G_K$, $R_K$ description.

The fluctuation $\Delta k$ may be positive or negative. The transitions we consider are transitions of electrons. On the average a constant number makes the transition from $K$ in unit time. That constant number is determined by $\bar{G}$. But the number that is leaving in a specific short interval is somewhat higher or lower than the average because of $s\Delta G$. Therefore, the number of electrons present in $K$ fluctuates too. If too many electrons are leaving $K$, $\Delta k = -\beta = -\Delta n$, if too few are leaving, $\Delta k = +\beta = -\Delta n$, where $\beta$ is a positive number.

A positive fluctuation in $K$ may decay because of two mechanisms:

$\Delta k$: A positive number $\beta$ of electrons leaves the center and goes to the band: $\Delta k = -\beta$.

$\Delta n$: A negative number $-\gamma$ of electrons goes from the band to $K$, and changes $K$ with the amount $\Delta k = -\gamma$.

Whether in the initial stage, with $\Delta k \approx \Delta n$, the $\Delta k$ or the $\Delta n$ process dominates, depends on the ratio $n/k^*$, as is shown by (31).

If $n \gg k^*$ the $\Delta k$ process dominates. The electrons in the center have only one place to go to: the band, $\Delta k = \Delta n$.

If $n \ll k^*$ the $\Delta n$ transition dominates. The $\Delta n$ electrons in the band will redistribute themselves over all centers: $-\Delta n = \Delta a + \Delta b + \cdots + \Delta k + \cdots$ All terms on the right-hand side have the same sign. Only a very small fraction goes to $K$, so $\Delta k$ hardly changes.

The two situations are presented in Fig. 3.

4.1. The kinetics when $n \ll x_i^*$

In this section the condition $n \ll x_i^*$ is used, which is stricter than the condition for mixing, $n \ll \Sigma x_i^*$ (Section 3). This makes it possible to sketch a very simple picture of the kinetics of the decay of the fluctuations: $\Delta n$ processes dominate over the $\Delta k$ processes.

In the short time interval $0 < t < \delta t$, the fluctuations $\Delta n = \beta$ and $\Delta k = -\beta$ are generated. (see Fig. 3). During the decay the $\Delta n$ transition dominates over the $\Delta k$ transition, since $n \ll k^*$ and $\Delta n = -\Delta k$ (31). The $\beta$ electrons in the band redistribute themselves over all centers with transition rates $\beta \theta_A, \beta \theta_B, \cdots \beta \theta_K$. So $\Delta n$ decays very fast with

$$\theta_1 = \theta_A + \theta_B + \cdots + \theta_K + \cdots.$$  \hspace{1cm} (32)

Only a small fraction, $\beta \theta_K/\theta_1$, goes to $K$ level, from where hardly any electron returns to the band with $\mu \theta_K$. This means that $\Delta k$ practically stays $-\beta$.

Now consider the situation for $t \gg \theta_1$. Left of $K$ there are $(r_k - 1)$ fast centers in equilibrium with the band $\sum \gamma_i \Delta x_i = \beta$. For each center equilibrium implies $\Delta x_i/x_i^* = \Delta n/n$. Summing
the results for the $(r_k - 1)$ centers left of $K$: 
$(r_k - 1)\Delta n = \mu \Sigma x_j = \mu \beta$ resulting in \( \Delta n = \mu / (r_k - 1) \beta \). Right of $K$ all levels are practically empty. Left of $K$ there is an excess of $\beta$ electrons from the band; in $K$ there still is the original deficit $-\beta$. These fluctuations $\beta$ and $-\beta$ decay extremely slowly; the electrons go from the traps—via the band—to the $K$ center. The transitions between traps and the band are fast, ensuring a permanent equilibrium: Then two processes between band and $K$ follow

$$
\frac{d\Delta n}{dt} = -\left( \frac{\mu \beta}{r_k - 1} \right) \theta_K + (-\beta)(\mu \theta_K)
$$

$$
= -\left( \frac{r_k}{r_k - 1} \right) \mu \beta \theta_K \approx \mu \beta \theta_K.
$$

(33)

In first approximation the lines for $\Delta a$, $\Delta b$, etc. and $\Delta k$ in Fig. 3 are horizontal; after further consideration they very slowly go to zero with

$$
\theta = \left( \frac{r_k}{r_k - 1} \right) \mu \theta_K \approx \mu \theta_K.
$$

(34)

4.2. The kinetics when $n \gg \Sigma x_i$

In the time interval $0 < t < \delta t$ the fluctuations $\Delta n = \beta$ and $\Delta n = -\beta$ are generated. During the decay the $\Delta k$ transition dominates, since $n \gg k*$ and $\Delta n = -\Delta k$ (31). As a consequence

$$
\theta = \theta_K
$$

(35)
as follows from
\[
\frac{d\Delta n}{dt} = -\beta (v_0 k) + (-\beta)\theta_K \approx -\beta \theta_K. \quad (36)
\]

The noise spectrum of the McWhorter model is the summation of the characteristic spectra of \( A, B \), etc. Since \( g(\tau) \propto 1/\tau \) (9) and since the variances are equal: \( x_i^* \) (16), the McWhorter model produces \( 1/f \) noise under this condition, \( n \gg \sum x_i^* \).

In the Sections 4.1 and 4.2, and especially in Fig. 3, it was explained how \( n \ll x^* \) leads to mixing and \( n \gg \sum x^* \) to addition. The crude approximations used led to surprisingly good results even for details like the slow relaxations with \( \theta_d = \mu \theta_K \) (34), as will be shown later (45). More detailed calculations are presented in the following sections. The results are complicated and, therefore, less easy to visualize, but they show that relations like (34) are essentially correct. They also show that the results obtained here under the condition \( n \ll x^* \) can be extended into the whole region of mixing where \( n \ll \Sigma x^* \) holds.

5. More detailed calculations for mixing with \( n \ll x_i^* \)

5.1. The eigenvalues \( \theta_k \)

We consider a McWhorter model with a band and \( r_z \) centers, \( A, B, \ldots \) \( Z \). The indices are defined in such a way that \( \theta_A > \theta_B > \cdots > \theta_Z \). We start with the approximation \( \theta_A \gg \theta_B \gg \cdots \gg \theta_Z \).

For each center \( K \) there is an equation for the \( G_K \) and \( R_K \) transitions
\[
\frac{d\Delta k}{dt} = -\theta \Delta k - \Delta G + \Delta R = -\mu \theta_K \Delta k + \theta_K \Delta n,
\]
\( \mu \theta_K - \theta \Delta k - \theta_K \Delta n = 0. \quad (37) \)

There are \( (r_Z + 1) \) unknowns \( \Delta a, \Delta b, \ldots, \Delta z \) and \( \Delta n \) and \( (r_Z + 1) \) equations: \( r_Z \) equations like (37) and
\[
\Delta a + \Delta b + \cdots + \Delta k + \cdots + \Delta z + \Delta n = 0. \quad (38)
\]

We find non-trivial solutions for \( \Delta a \ldots \Delta n \) if the determinant (39) is zero
\[
\begin{vmatrix}
(\mu \theta_A - \theta) & 0 & 0 & \cdots & 0 & -\theta_A \\
0 & (\mu \theta_B - \theta) & 0 & \cdots & 0 & -\theta_B \\
0 & 0 & (\mu \theta_C - \theta) & \cdots & 0 & -\theta_C \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (\mu \theta_Z - \theta) & -\theta_Z \\
1 & 1 & 1 & \cdots & 1 & 1
\end{vmatrix} = 0.
\]

There are \( r_Z \) values of \( \theta \), which we call \( \theta_1 > \theta_2 > \cdots > \theta_r \) \( (r_A = 1, r_B = 2 \cdots r_k = r_k, \cdots, r_z = r_z) \), \( \theta_1 \) gives a solution for \( \Delta a, \Delta b \cdots \Delta n \), which we call \( \Delta a_1, \Delta b_1 \cdots \Delta n_1 \). A fluctuation \( \Delta a \) decays as the sum of \( z \) exponentials
\[
\Delta a(t) = \Delta a_1(t) + \Delta a_2(t) + \cdots = \Delta a_1(0) e^{-\theta_1 t} + \Delta a_2(0) e^{-\theta_2 t} + \cdots. \quad (40)
\]

We approximate the determinant (39) by neglecting the smaller term in the diagonal elements \( (\mu \theta_K - \theta) \), so that the diagonal elements become either \( \mu \theta_K \) or \( -\theta \).

1. We start with the question “Do \( \theta \)-values exist that are larger than \( \mu \theta_K \)?” Such a value, called \( \theta_1 \), is automatically larger than any member of the set \( \mu \theta_A, \mu \theta_B \cdots \mu \theta_Z \). The set of Eqs. (37) then becomes
\[
-\theta_1 \Delta a = \theta_A \Delta n \\
\vdots \\
-\theta_1 \Delta z = \theta_Z \Delta n
\]

The sum of these \( r_z \) equations gives
\[
-\theta_1 [\Delta a + \cdots + \Delta z] = [\theta_A + \cdots + \theta_Z] \Delta n.
\]

Because of (38)
\[
\theta_1 = \sum_{i=1}^{r_z} \theta_i. \quad (41)
\]

There is one solution, \( \theta_1 \), larger than \( \mu \theta_A \). The same result was obtained from the kinetic model (32).

2. Therefore, we return to the kinetic model for a first guess at the other \( \theta \)-values. Eq. (34) suggests \( \theta = \mu \theta_K \). Naturally, we are not very certain about the factor 1, let us try, therefore, \( \theta_{r_k} = x \mu \theta_K \) where \( x \) is an unknown constant, close to 1.
Therefore we ask ourselves: “Are there $\theta$-values between $\mu \theta J$ and $\mu \theta L$?”

In the diagonal elements of (39) we neglect all $\mu \theta J \cdots \mu \theta J$ and use $\mu \theta J_k$ there. We keep the complete element $(\mu \theta K - \chi x \mu \theta K)$, and for all other diagonal elements we write $-\chi x \mu \theta K$. The set of Eqs. (37) becomes

$$
m \mu \theta J \Delta a_k = \theta A \Delta n_k$$

$$
\vdots
$$

$$
\mu \theta K \Delta l_k = \theta Z \Delta n_k
$$

$$
(1 - x) \mu \theta K \Delta r_k = \theta K \Delta n_k \rightarrow \Delta r_k = \frac{1}{1 - x} \mu \Delta n_k,$$

$$
x \mu \theta K \Delta l_k = \theta L \Delta n_k
$$

$$
\vdots
$$

$$
x \mu \theta K \Delta z_k = \theta Z \Delta n_k
$$

$$
\Delta a + \cdots + \Delta z = [\Delta a + \cdots + \Delta J] + \Delta k + [\Delta l + \cdots + \Delta z] = -\Delta n \approx 0,
$$

$$
(r_k - 1) \mu \frac{\Delta n}{\delta t} + \frac{1}{1 - x} \mu \Delta n + [0] = 0,
$$

$$
x = \frac{r_k}{r_k - 1},
$$

$$
\theta_{r_k} = x \mu \theta K = \frac{r_k}{r_k - 1} \mu \theta K \approx \mu \theta K.
$$

For any $r_k \geq 2$ there is one $\theta_{r_k}$ value given by (45). The result of the kinetic model (34) $\theta \approx \mu \theta K$ agrees with (45).

(3) The last question is “Do $\theta$-values exist that are smaller than $\mu \theta Z$?” Such a value would require

$$
\mu \theta J \Delta A a = \theta A \Delta n
$$

$$
\vdots
$$

$$
\mu \theta Z \Delta z = \theta Z \Delta n
$$

The sum gives $\mu(\Delta a + \cdots + \Delta z) = r_z \Delta n$. Since $\mu \ll 1$ and $r_z \ll 1$ Eqs. (51) are contradictory for $\theta < \theta_z$.

In conclusion: There are $r_z$ solutions $\theta$ of (39).

$$
\theta_1 = \sum_{i=1}^{r_z} \theta_i.
$$

$$
(45) \quad (r_z - 1) \text{ solutions}
$$

$$
\theta_{r_k} = \frac{r_k}{r_k - 1} \mu \theta K \approx \mu \theta K, \quad 2 \leq r_k \leq r_z.
$$

5.2. Variances

Using the results (41) and (45) we will now recalculate the variances. The agreement between the variances from (41) and (45) and from the thermodynamic discussions of Section 3 give additional support to our simple kinetic interpretation of the mathematical conditions $n \ll \Sigma k^*$ and $n \gg \Sigma k^*$.

We start by considering the development of the number of electrons in a chosen level $K$. If we observe the situation in $K$ for an extended period, we find the variance $(\Delta k)^2$ from $T/\delta t$ uncorrelated contributions (26), each decaying with the lifetime $\tau_{\text{life}}$.

$$
(\Delta k)^2 = \frac{T}{\delta t} \frac{1}{T} \int_T^0 [\Delta (\Delta k)]^2 \left[e^{-t/\tau_{\text{life}}}ight]^2 dt = \frac{T}{\delta t} \frac{1}{T} \int_T^0 2R_K \delta t e^{-2t/\tau_{\text{life}}} dt = R_K \tau_{\text{life}}.
$$

(47)

If we consider a sample with only one type of trap, the characteristic $K$ spectrum will be found with $\tau_{\text{life}} = \tau_K$. The variance follows from (47), (3), and $n \ll k^*$.

$$
(\Delta k)^2 = R_K \tau_K = n.
$$

(48)

In agreement with Part 1 Eq. (11) [1]. If we consider trap $K$ in a MeWhorter model $(n \ll k^*)$ relation (45) gives $\tau_{\text{life}} = \tau_k = n/\mu$.

The variance of the fluctuations “from $K$” is

$$
(\Delta k)^2 = \frac{1}{\mu} R_K \tau_K = \frac{1}{\mu} n = k^*.
$$

(49)

In the time interval $0 < t < \delta t$ the number of electrons created in $K$ is $\Delta (\Delta k)$, (25). At $t > \tau_k$ “nothing” has changed in $K$, while the free electrons in the band have decayed with $\tau_k$ to the traps (Few of them fell into $K$). The fast traps—left of $K$—are now in equilibrium with the band; the slow traps—right of $K$—are practically empty. This is the situation given by (42)–(45)

$$
\Delta (\Delta a) = \cdots = \Delta (\mu \theta) = - \frac{1}{r_k - 1} \Delta (\Delta k) = \frac{1}{\mu} \Delta (\mu \theta).
$$

(50)
All these fluctuations decay with $\theta_{rk} \approx \mu \theta_k$. 
Decaying with $\theta_{rk}$ is synonymous with “from $K$”. In our notation: $\delta x_{rk} \equiv K \delta x$

In analogy to (49) we find $\bar{\chi}(\Delta x)^2 = x^*$. Since $a^* = b^* = \cdots = z^*$ it follows that

$$\bar{\chi}(\Delta x)^2 = x^* = k^* = \kappa(\Delta k)^2.$$

There are small contributions of many $\chi(\Delta k)^2$. According to (50)

$$\bar{\chi}(\Delta k)^2 = \left(\frac{1}{r_x-1}\right)^2 \chi(\Delta x)^2 \text{ if } r_x > r_K$$

$$= 0 \text{ if } r_x < r_K$$

The contribution “from $X$” is

$$\bar{\chi}(\Delta k)^2 = \left(\frac{1}{r_x-1}\right)^2 \chi(\Delta x)^2$$

$$= \left(\frac{1}{r_x-1}\right)^2 \kappa(\Delta k)^2 \ll \kappa(\Delta k)^2.$$  

(53)

In good approximation: There is no contribution to $\kappa(\Delta k)^2$ other than “from $K$”. The variance of $\Delta k$ follows from (49)

$$\kappa(\Delta k)^2 \approx \kappa(\Delta k)^2 = k^*.$$

In agreement with (16).

The number of free electrons generated in $0 < t < \delta t$ is the sum of contributions from $A$ ... from $Z$. Consider the contribution from $K$. In analogy to (26)

$$\bar{\chi}(\Delta k \delta m)^2 = 2R_K \delta t.$$

(54)

These electrons decay from the band to the traps with $\theta_1 = \theta_A + \cdots + \theta_Z$ (see Fig. 3). Their lifetime in the band is $\tau_{\text{life}} = 1/\theta_1$. The variance, averaged over an extended period, is, in analogy to (47)

$$\bar{\chi}(\Delta n_k)^2 = R_K \tau_{\text{life}} = R_K \tau_{\text{life}} \tau_{11} = \frac{n K}{\theta_1}.$$  

(55)

The variance of $\Delta n_k$, coming from all the traps $A ... Z$ is

$$\bar{\chi}(\Delta n_k)^2 = \sum_k n \frac{\theta_k}{\theta_1} = n.$$  

(56)

According to (50) there are contributions $[\chi(\Delta n_k)]^2$ to the variance of $n$ from each level $K$:

$$[\chi(\Delta n_k)]^2 = \frac{\mu^2}{(r_k-1)^2} [\chi(\Delta k)]^2$$

$$= \frac{\mu^2}{(r_k-1)^2} 2 R_K \delta t,$$

(57)

$$[\chi(\Delta n_k)]^2 = \frac{\mu^2}{(r_k-1)^2} R_K \tau_{rk}$$

$$= \frac{\mu}{(r_k-1)^2} R_K \tau_{rk} = \frac{\mu}{(r_k-1)^2} n \ll n.$$  

(58)

The variance of $n$ is

$$\bar{\chi}(\Delta n)^2 = (\Delta n_1)^2 + \sum_k (\Delta n_k)^2$$

$$= n \left[1 + \sum_k \frac{\mu}{(r_k-1)^2}\right] \approx n$$  

(59)

In conclusion: the variances $\bar{\chi}(\Delta k)^2 = k^* \text{ (49)}$ and $\bar{\chi}(\Delta n)^2 = n \text{ (59)}$ agree with the results from the thermodynamic considerations (16) and (17). This agreement demonstrates that we can obtain reliable results by our analysis of the contributions “from $A$” to “from $Z$”. So we feel confident to go one step further: calculation of the spectra.

5.3. Spectra

The only spectra we can observe are the spectra of fluctuations in the number of free electrons.

(1) The majority of the free electrons decay with $\tau_1$. They produce the spectrum

$$S_{\text{f1}}(f) = \frac{4(\Delta n_1)^2 \tau_1}{1 + (2\pi f \tau_1)^2} = \frac{4n \tau_1}{1 + (2\pi f \tau_1)^2},$$

(60)

which is white up to high frequencies of the order $\theta_1$.

(2) There is a small contribution to $n : \Delta n_k$ from each of the levels. Judging from the variances alone, we would expect that these conditions can be neglected, as is suggested by (58). However, the factor $\tau_{rk}$ in the numerator of $S_{\text{f1}}(f)$ leads to surprises: The white branch of the spectrum
becomes
\[ S_{r_k}(\text{white}) = 4 \frac{\mu}{(r_k - 1)^2} \eta^{-\tau_K} \]
\[ = \frac{1}{(r_k - 1)^2} \theta_1 \{ 4 \pi \sigma_1 \}. \] (61)

Because of (58) and (45),

The \( S_{r_k} \) spectrum can be observed if its white branch is higher than the white \( S_{n_1} \) spectrum (60)
\[ 1 \left( \frac{\theta_1}{\theta_K} \right) \gg 1, \] (62)

which is the case for most of the \( r_k \) spectra. If the white branches are visible, the group of \( S_{r_k} \) spectra constitutes a “1/\( f \)-like” spectrum. The factors \( \theta_1/\theta_K \) provide the required \( g(\tau) \propto 1/\tau \) statistical weight. The factor \( (r_k - 1)^2 \) slowly varies over a wide frequency range; thereby changing the spectrum from 1/\( f \) to 1/\( f \)-like.

5.4. Less rigorous approximations

(1) A better approximation for \( R/k* \) and \( R/n \) follows from the definition of \( \theta_k \) (3).
\[ \theta_K = R \left( \frac{1}{n} + \frac{1}{k*} \right) = R \left( \frac{1}{n} + \frac{1}{k*} \right) = R \left( 1 + \mu \right). \] (63)

From which it follows that
\[ \frac{R}{n} = \frac{\theta_K}{1 + \mu}, \quad \frac{R}{k*} = \frac{\mu}{1 + \mu} \theta_K \] (64)

instead of \( \theta_k \) and \( \mu \theta_k \), respectively.

(2) According to the definitions (10)–(12) of the McWhorter model there is a relation between the characteristic parameters \( \theta \),
\[ \theta_K = \theta_0 e^{-\eta(r_k - r_i)}. \] (65)

In the present approximation we keep the exponentials, whereas before we had chosen for them to be equal to \( \infty \) or 0, if \( r_k \neq r_i \). We expect \( \theta_0 \) to be close to \( \mu \theta_k \), so we again introduce \( x_k \)
\[ \theta_{r_k} = x_k \mu \theta_K. \] (66)

The processes to and from the trap are now described by
\[ \frac{d \Delta i}{dt} = -\theta_{r_k} \Delta i = -\Delta G_i + \Delta R_i \]
\[ = \frac{-\mu}{1 + \mu} \theta_0 \Delta i + \frac{1}{1 + \mu} \theta_0 \Delta n, \] (67)

which analogous to (37), gives
\[ \mu \left[ 1 - x(1 + \mu) \theta_0 \right] \Delta i = \Delta n, \] (68)
\[ \Delta i = \frac{1}{n - x(1 + \mu)e^{-\eta r_i} - \mu} \Delta n \]
\[ = \frac{1}{1 - ce^{-\eta r_i} \mu} \Delta n \] (69)
with \( c \equiv x(1 + \mu)e^{-\eta r_k} \).

Summation over terms, as presented in Eq. (69), can be replaced by integrals, where we use
\[ \int \frac{1}{1 - ce^{-\eta r_i}} dr_i = r_i - \frac{1}{\eta} \log \{ \pm (1 - ce^{-\eta r_i}) \} \]

+ if \( (1 - ce^{-\eta r_i}) > 0 \)
- if \( (1 - ce^{-\eta r_i}) < 0 \). (70)

We calculate new expressions for the three groups of terms in (42). (see Fig. 2 for the limits of integration)
\[ \frac{\Delta a + \cdots + \Delta j}{\mu \Delta n} = \sum_{r_i = 1}^{r_k-1} \frac{1}{1 - ce^{-\eta r_i}} \int_{1/2}^{r_k-1/2} \frac{1}{1 - ce^{-\eta r_i}} dr_i \]
\[ = (r_k - 1) - \frac{1}{\eta} \log \left\{ 1 - x(1 + \mu)e^{-\eta/2} \right\} \]
\[ = (r_k - 1) + F, \] (71)
\[ \frac{\Delta k}{\mu \Delta n} = \frac{1}{1 - x(1 + \mu)}. \] (72)
\[ \frac{\Delta l + \cdots + \Delta z}{\mu \Delta n} = \sum_{r_i = 1}^{r_k-1} \frac{1}{1 - ce^{-\eta r_i}} \int_{r_i+1/2}^{r_k-1} \frac{1}{1 - ce^{-\eta r_i}} dr_i \]
\[ \approx - \frac{1}{\eta} \log \left\{ x(1 + \mu) - e^{-\eta/2} \right\} = - G. \] (73)
The results (71)–(73) are substituted in (38)
\[ \{\Delta a + \cdots + \Delta j\} + \Delta k + \{\Delta l + \cdots + \Delta z\} = -\Delta n, \]
(74)
\[
\left[ (r_k - 1) + F \right] + \frac{1}{1 - x(1 + \mu)} + \{-G\} \\
\times \frac{1}{\mu} \Delta n = -\Delta n,
\]
(75)
\[ (r_k - 1) + F - \frac{1}{1 - x(1 + \mu)} - G = -\mu. \]
(76)

1. If \( \eta > 1, e^{-\eta^2} \rightarrow 0 \). This results in \( F = 0, G = 0 \). If \( \mu \ll 1 \) we should find the old result (44) and (45). Indeed, we do:
\[ (r_k - 1) + \frac{1}{1 - x} = 0 \Rightarrow x = \frac{r_k}{r_k - 1}, \]
\[ \theta_{rk} = \frac{r_k}{r_k - 1} \mu \theta_K \approx \mu \theta_K. \]
(77)

2. The influence of \( \eta \) on \( F \) and \( G \) is much stronger than the influence of \( x \) and \( \mu \). Therefore we take \( x = 1, \mu = 0 \). In this approximation
\[ F = G = \frac{1}{\eta} \log(1 - e^{-\eta^2}). \]
(78)

If \( \eta \) is of the order 1, then \( F \) is of the order 0.1. More precisely: if \( 0.2 < \eta < 2 \), then \( 2 < F < 0.03 \). This gives small corrections to the results of the rigorous approximation. (42)–(45) There are corrections but only close to \( r_k \). The corrections are positive for \( r_i < r_k \). There are small negative contributions now for \( r_i > r_k \). The sums over the two groups have no influence on the values of \( x \) and \( \theta \), since in Eq. (76) \( F = G \).

6. More detailed calculations for addition with \( n \gg \Sigma x_i^* \)

6.1. Eigen values \( \theta_{rk} \) and spectra

From the definition of \( \theta_K \) (3) it follows that
\[ \theta_K = R_K \left( \frac{1}{k^*} + \frac{1}{n} \right) = \frac{R_K}{k^*} (1 + v), \quad v = \frac{k^*}{n}. \]
(79)
\[ R_K = \frac{1}{1 + v} \theta_{rk}, \quad \frac{R_K}{n} = \frac{v}{1 + v} \theta_K \approx v \theta_K. \]
(80)

Later (in Eq. (86)) it will become clear that it is essential to keep the term \( v \), however small, in the denominator. At each center \( K \), there is a relation for \( \Delta k \) and \( \Delta n \) (compare 37).
\[
\frac{d\Delta k}{dt} = -\Delta k = -\Delta G_K + \Delta R_K \\
= \frac{G_K}{k^*} \Delta k + \frac{R_K}{n} \Delta n \\
= \frac{-1}{1 + v} \theta_{rk} \Delta k + \frac{v}{1 + v} \theta_K \Delta n, \]
(81)
\[
\left(1 + (1 + v)\frac{\theta}{\theta_K}\right) \Delta k = v \Delta n. \]
(82)

We assume again \( \theta_{rk} = x \theta_K \) with \( x \approx 1 \). In the rigorous approximation the smaller term between the square brackets in Eq. (82) is neglected if \( r_i \neq r_k \):

\[
\begin{align*}
1 - x(1 + v) \frac{\theta_{rk}}{\theta_{rk}} \Delta a &\approx \Delta a = v \Delta n \\
&\rightarrow \Delta a = v \Delta n \\
&\vdots \\
1 - x(1 + v) \frac{\theta_{rk}}{\theta_{rk}} \Delta j &\approx \Delta j = v \Delta n \\
&\rightarrow \Delta j = v \Delta n \\
1 - x(1 + v) \Delta k &\approx [1 - x(1 + v)] \Delta k = v \Delta n \\
&\rightarrow \Delta k = \frac{1}{1 - x(1 + v)} \Delta n. \quad \text{(83)}
\end{align*}
\]
\[
\begin{align*}
1 - x(1 + v) \frac{\theta_{rk}}{\theta_{rk}} \Delta l &\approx \frac{\theta_{rk}}{\theta_{L}} \Delta l = v \Delta n \\
&\rightarrow \Delta l = \frac{\theta_{rk}}{\theta_{L}} v \Delta n \approx 0 \\
&\vdots \\
1 - x(1 + v) \frac{\theta_{rk}}{\theta_{rk}} \Delta z &\approx \frac{\theta_{rk}}{\theta_{Z}} \Delta z = v \Delta n \\
&\rightarrow \Delta z = -\frac{\theta_{rk}}{\theta_{Z}} v \Delta n \approx 0
\end{align*}
\]
Summing over all levels:
\[
\{\Delta a + \cdots + \Delta j\} + \Delta k + \{\Delta l + \cdots + \Delta z\} = \left[ (r_k - 1) + \frac{1}{1 - x(1 + v)} + 0 \right] v\Delta n = -\Delta n. \tag{84}
\]
From which it follows that
\[
x = 1 \quad \theta_{rk} = \theta_K. \tag{85}
\]
From (83) and (85)
\[
\Delta k - \frac{1}{1 - 1 - v} v\Delta n = -\Delta n. \tag{86}
\]
Summarizing:
\[
\Delta a = \cdots = \Delta j = v\Delta n \approx 0, \quad \Delta k = -\Delta n, \quad \Delta l = \cdots = \Delta z = 0. \tag{87}
\]
Since \(\theta_{rk} = \theta_K\) (85) the original characteristic spectra can be added together, without any mixing. For each level \(K\) the fluctuations are \(\Delta k = -K\Delta n\), independent of fluctuations in other levels. There are no correlations between fluctuations of different levels.

There is an unimportant complication. As long as \(\Delta n\) differs from zero some electrons will be trapped in the \(A, B \ldots\) centers (see Fig. 3). Since these traps are fast, equilibrium between \(A\) and the band will be established in a time that is short compared to \(\tau_K\). From then on \(\Delta a = v\Delta n\). Therefore the number of free electrons \(\Delta n\) is not exactly \(-\Delta k\), but it is a large, constant fraction thereof.

\[
\Delta n(t) = [1 - (z_k - 1)v](-\Delta k(t)) \approx -\Delta k(t). \tag{88}
\]

The influence of the correction \(1 - (r_k - 1)v\) will not be noticed in the spectrum “from \(K\)”, since the trapped fractions are small and equal in all traps with \(r < r_k\). The variances become
\[
(\Delta k)^2 = k^*, \tag{89}
\]
\[
(\Delta n)^2 = \sum_K (K\Delta n)^2 = \sum_K (\Delta k)^2 = \sum_K k^* \tag{90}
\]
in agreement with (16) and (18). The spectra are
\[
S_{nk}(f) = \frac{4k^*\tau_K}{1 + (2\pi f\tau_K)^2}, \tag{91}
\]
which add up to a \(1/f\) spectrum.

6.2. Less rigorous approximations

We now keep the exponentials, where in the rigorous approximation we took them to be \(\infty\) or 0:
\[
(\Delta i) = \frac{1}{1 - x(1 + v)e^{-\eta r_k}e^{-\eta r_l}} v\Delta n = \frac{1}{1 - ce^{\eta r_k}} v\Delta n \tag{92}
\]
with \(c = x(1 + v)e^{-\eta r_k}\).

The three groups of (83) and (84) now become
\[
(1) \quad \Delta a + \cdots + \Delta j \quad \frac{1}{v\Delta n}
\]
\[
(2) \quad \Delta k \quad \frac{1}{1 - x(1 + v)}, \tag{94}
\]
\[
(3) \quad \Delta l + \cdots + \Delta z \quad \frac{1}{v\Delta n}
\]
\[
= \sum_{r_k + 1}^r \frac{1}{1 - ce^{\eta r_k}} \int_{r_k + 1/2}^{r_k - 1/2} \frac{1}{1 - ce^{\eta r}} dr
\]
\[
= (r_k - 1) - \frac{1}{\eta} \log \{1 - x(1 + v)e^{-1/2\eta}\} \tag{93}
\]
\[
= (r_k - 1) + F, \tag{93}
\]
\[
(2) \quad \frac{\Delta k}{v\Delta n} = \frac{1}{1 - x(1 + v)}, \tag{94}
\]
\[
= \sum_{r_k + 1}^r \frac{1}{1 - ce^{\eta r_k}} \int_{r_k + 1/2}^{r_k - 1/2} \frac{1}{1 - ce^{\eta r}} dr
\]
\[
= \frac{1}{\eta} \log \{x(1 + v) - e^{-1/2\eta}\} = -G. \tag{95}
\]

The influence of \(F\) and \(G\) is the same as in the case of mixing. That situation was analyzed around relation (78). Now there are the same small corrections close to \(r_k\).

7. Discussion of the results and application to the \(1/f\) noise of MOSTs

7.1. The \(\Delta n, \Delta \mu\) problem

In the previous sections we discussed what kind of spectrum results from many simultaneous generation-recombination processes. We concentrated on the McWhorter model, where the \(GR\)-processes have different relaxation times with statistical weights \(g(\tau) \propto 1/\tau\). We showed that a \(1/f\) spectrum results if the characteristic spectra of
the different GR-processes have the same variance and—most importantly—if they may simply be added. Additivity is the main subject of this and the previous publication [1]. A useful conclusion is that the McWhorter model produces the naively expected 1/f noise only if $n \gg \Sigma x_i$.

Since the McWhorter model discusses $GR$-noise, the resulting 1/f noise is a fluctuation in the number of free electrons. The question then arises whether such a model of number fluctuations is applicable at all. It has experimentally been proved that in many cases 1/f noise is a fluctuation in the mobility. The experimental evidence followed from studies of 1/f noise in thermoelectricity, the Hall effect and conductivity with two scattering mechanisms [6–8].

A weak point for our present discussion is that all this experimental evidence was found with homogeneous samples of high-quality material. In principle it is possible that the 1/f noise found in MOSTs is not the same as the mobility 1/f noise in homogeneous samples. Whether the McWhorter model is of any relevance to the MOSTs can only be answered by experiments of the type that have been performed on the homogeneous samples, and that should prove that in MOSTs the 1/f noise is a fluctuation in number. Such experiments have never been done. There has been ample discussion on number fluctuations versus mobility fluctuations in MOSTs [9]. But such discussions have not been decisive. Some effects can be explained by either of the models, but neither model can definitely be excluded.

We cannot consider the MOST as an unimportant complication of a situation that otherwise is perfectly clear. Too many noise studies have been done on MOSTs, where at first sight the McWhorter model gives the right results. Many people do not bother about the theoretical correctness, because they see that the model works well. For a final judgment on the validity of the McWhorter model we need an experimental determination of the nature of 1/f noise ($\Delta n$ or $\Delta \mu$): However, there is one aspect of the model that we can investigate now. Granted, the McWhorter model explains the shape of the spectrum is 1/f, but does the model predict the right magnitude? In a recent publication [4] Van Vliet concluded that because of the correct shape of the spectrum “the McWhorter model can hold”. However, she did not give an estimate for the magnitude. Her condition for a 1/f spectrum (following her Eq. 2.18) is analogous to our conclusion for additivity, mixing is not discussed by her.

We express the 1/f noise in the usual way as

$$\frac{S}{n^2} = \frac{f}{nf}.$$  \hspace{1cm} (96)

We use the notation introduced in these publications on additivity, which differs somewhat from the usual one. $n, a = \text{total number of free electrons}$, trapped electrons in the whole sample. $N, A = \text{total number of states in the band}$, in the traps.

7.2. Calculation of $\alpha$ in a McWhorter model

For the calculation of $\alpha$ we need numerical values of numbers of states, electrons, etc. The literature on MOSTs from the past 15 years shows a reasonable agreement concerning numerical values of trap concentrations and of effective number of states in the conduction band. All calculations will be made for a MOST with an area of 1 cm$^2$.

The effective number of states in the conduction band is determined by the effective thickness of the conducting channel in Si. This thickness is assumed to be 100 Å = $10^{-6}$ cm, so that $N$, the number of states, is $N = 10^{-6} \times 10^{19} = 10^{13}$. The traps far from the interface have very large $\tau$-values because of the exponential dependence on $x$, being the distance from the interface. These extremely slow states have no influence, since no transitions occur during the measuring time. The measuring time determines an effective thickness $t_{\text{ox}}$. One usually assumes $t_{\text{ox}} = 30$ Å = $3 \times 10^{-7}$ cm. The simple model of Fig. 1 has traps, all with the same energy $E_{\text{trap}}$. In modern publications one assumes a homogeneous trap distribution across the band gap of the oxide. The average value of the bulk concentration is about $2 \times 10^{17}$ cm$^{-3}$ eV$^{-1}$, for the surface concentration one finds $5 \times 10^{10}$ cm$^{-2}$ eV$^{-1}$. With $t_{\text{ox}} = 3 \times 10^{-7}$ cm the absolute number in a MOST of 1 cm$^2$ becomes
$5 \times 10^{10}$ eV$^{-1}$. The Fermi level in the oxide is at a position determined by the conditions in the silicon conductive layer. The number of traps to be considered are within an energy band of $kT$ around the Fermi level. For the numbers of McWhorter traps this gives $M = (1/40) \times 5 \times 10^{10} = 10^9$. A more detailed analysis of the window around the Fermi level has been given by Van Vliet [4]. In the simple model of Fig. 1 we take all traps within 0.6 eV to be concentrated at one single energy level $E_{\text{trap}}$. The number of McWhorter states then is $M = 0.6 \times 5 \times 10^{10} = 3 \times 10^{10}$.

In Fig. 2 we choose finite elements with thickness $\xi$. As a result the discussion of the kinetics became simple. We can easily determine whether fluctuations in a trap decay by transitions to and from the trap. In Fig. 3 the transitions to and from $K$ can be compared with the transitions left of $K$ and those right of $K$. The use of $\xi$ makes the physical interpretation easier, and it enables us to present the useful Fig. 3.

However, the expressions for the final results contain $\xi$, which could be chosen freely; whereas the final results cannot depend on $\xi$. So we have to eliminate $\xi$. Another difficulty is that when making $\xi$ extremely small, there are no situations with $n \ll x_i$. This is the condition used in the discussions of the kinetics of mixing. It leads to the simple pictures of Fig. 3. But the algebraic calculations in Section 5.1 were done using $n \ll \Sigma x_i$, which is independent of $\xi$.

The total number of oxide traps in the McWhorter model is $M$; $m$ is the number of trapped electrons.

$$ M = \sum_{i=1}^{r_x} X_i = r_x X_i, \quad m = \sum_{i=1}^{r_x} x_i = r_x x_i, $$

$$ \frac{1}{m^*} = \frac{1}{M - m} \frac{1}{r_x x_i} + \frac{1}{r_x (X_i - x_i)} = \frac{1}{r_x X_i} = \frac{1}{r_x x_i} \to m^* = \Sigma x_i^*. \quad (97) $$

In the expressions derived for transitions “from $K$” we write

$$ k^* = \frac{\xi}{t_{\text{ox}}} m^*. \quad (98) $$

A factor $\mu/(r_k - 1)$ appears in the expressions for the variances. We replace this factor by

$$ \frac{\mu}{r_k - 1} \approx \frac{\mu}{r_k} = \frac{n}{k^*} \frac{\xi}{s_k} = \frac{n t_{\text{ox}} \xi}{m^*} \frac{t_{\text{ox}}}{s_k} \approx 2 \frac{n}{m^*}. \quad (99) $$

The latter approximation is necessary since $t_{\text{ox}}/s_k$ will appear in integrals that can only be handled if we replace each $s_k/t_{\text{ox}}$ by the average value $S_k/t_{\text{ox}} = 1/2$. One useful relation is

$$ \int_{t_1}^{t_2} \frac{1}{\tau} \mathrm{d}\tau = \ln \frac{t_2}{t_1} = \frac{t_{\text{ox}}}{\lambda} - \frac{\xi}{\lambda} \approx t_{\text{ox}} \equiv A. \quad (100) $$

$A$ is the number of segments in Fig. 2 if we take $\xi = \lambda$ or $\eta = 1$. In the numerical calculations and in Fig. 4 we use $A = 30$, corresponding to $t_{\text{ox}} = 30$ Å and $\lambda = 1$ Å.

### 7.2.1. Variance and spectra when $n \ll m^*$

$$ (k \Delta k)^2 = \frac{1}{\mu} R_k \tau_{r_k} = \frac{1}{\mu} R_{k^*} \tau_{r_k} = \frac{1}{\mu} n = k^*. \quad (101) $$

This $k \Delta k$ decays with $\tau_{r_k}$. The spectrum is given by

$$ S_k = 4k^* \frac{\tau_{r_k}}{1 + (\omega \tau_{r_k})^2}. \quad (102) $$

The variance of the corresponding $k \Delta n$ is given by

$$ (\Delta n)^2 = \sum_{r=2}^{r_x} \frac{1}{\xi m^*} \frac{2}{\xi} = \sum_{r=2}^{r_x} \frac{2 \xi m^*}{t_{\text{ox}}} = 4 \frac{n^2}{m^*} \frac{\xi}{t_{\text{ox}}} = 4 \frac{n^2}{m^*}. \quad (104) $$

This justifies the last step in the derivation of (59). The spectrum of the component of $n$ that decays with $\tau_{r_k}$ is given by

$$ S_n = 4 \left[ \frac{2 \xi m^*}{n} \right] k^* \frac{\tau_{r_k}}{1 + (\omega \tau_{r_k})^2}. \quad (105) $$
where the term between square brackets follows from (103).

A 1/f-like spectrum results from the summation of all $S_n$ spectra (105) with different $\tau_{rk}$ values, $\tau_1$ excluded. For $k^*$ we substitute in (105)

$$k^* = \frac{\partial k}{\partial \tau} = \frac{\partial k}{\partial \tau} \frac{\partial \tau}{\partial \tau} = m^* \frac{1}{t_{ox}} - \frac{1}{d \tau}. \quad (106)$$

The relations $\partial s/\partial \tau$ and $\ln(\tau_{rk}/\tau_1) = \Lambda$ are true both for $\tau_{rk} = \tau_K$ in the case of addition as for $\tau_{rk} = (1/\mu)\tau_K$ in the case of mixing.

$$S = \int_{\tau_2}^{\tau_1} \left[ 4 \left( \frac{n}{m^*} \right)^2 m^* \frac{1}{t_{ox}} \frac{1}{\tau} \right] \frac{\tau_{rk}}{1 + (\omega \tau_{rk})^2} d\tau = 4 \frac{n^2}{m} \frac{1}{\Lambda f}. \quad (107)$$

Using the definition (96) for $\alpha$ we find

$$\alpha = 4 \frac{n}{m^*} \frac{1}{A} \quad (n \ll m^*). \quad (108)$$

7.2.2. Variances and spectra when $n \gg m^*$

In the case of addition we have

$$\tau_{rk} = \tau_K \quad \text{and} \quad k(\Delta k)^2 = (\Delta n_{rk})^2 = k^*. \quad (109)$$

That makes it possible to write down the expression for the 1/f noise

$$S = \sum k(\tau)S(\tau) = \int_{\tau}^{\tau_1} \frac{\partial k}{\partial \tau} \frac{\partial S}{\partial \tau} S(\tau) d\tau = m^* \frac{1}{A f} \int_{\tau}^{\tau_1} S(\tau) d\tau = m^* \frac{1}{n A f}. \quad (110)$$

From which it follows that

$$\alpha = \frac{m^*}{n} \frac{1}{A} \quad n \gg m^*. \quad (111)$$

7.3. $\alpha$ as function of the Fermi level

The Fermi level in the oxide is determined by the conditions in the silicon conducting channel. Therefore, with the very large number of MOSTs studied, the oxide Fermi levels correspond to silicon Fermi levels that could be anywhere in

\[\text{ARTICLE IN PRESS}\]
the upperhalf of the band gap. (Results for p-type are presented as if they were for n-type). The \( a \)-values found experimentally are mostly between \( 10^{-4} \) and \( 10^{-6} \) [9].

Fig. 4 shows the results calculated with (108) and (111) for two McWhorter models.

In Fig. 4(a) we consider the less realistic model with one kind of trap with energy \( E_{\text{trap}} \), which is chosen at \(-0.3\) eV. The figure is simple enough to show what happens if another value is chosen. It is only if \( E_{\text{trap}} \approx 0\) eV that a constant value of \( a \) is found. We take \( N = 10^{13} \) and \( M = 3 \times 10^{10} \) as has been explained in Section 7.2.

In Fig. 4(b) we consider the more realistic model with traps homogeneously distributed in energy. At a fixed position of the Fermi level there are \( M = 3 \times 10^{10} \) relevant traps in an energy band with width \( kT \) around the Fermi level (explained in Section 7.2).

The \( a \) values in Figs. 4a and b vary so much that inaccurate numerical values cannot be blamed for this wide variation.

The final conclusion from the calculation of \( a \) is that the McWhorter model does not give a correct description of the 1/f noise in MOSTs.

Acknowledgements

I wish to express my sincere thanks to Dr. L.K.J. Vandamme. Without his cooperation, Section 7 could not have been written. Thanks are also due to Dr. Ignace Hooge and Jenny van Oudbroekhuizen for their assistance in the preparation of the final version of this paper.

References